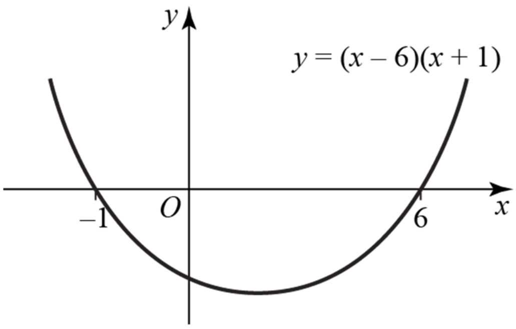
Exercise 1A

1 a $x^2-5x-6<0$ subtracting $5x+6$ from both sides

$(x-6)(x+1)<0$ factorising

So the critical values are $x=-1$ or 6

A sketch of $y=(x-6)(x+1)$ is



The solution corresponds to the section of the graph that is below the x-axis.
So the solution is $-1<x<6$

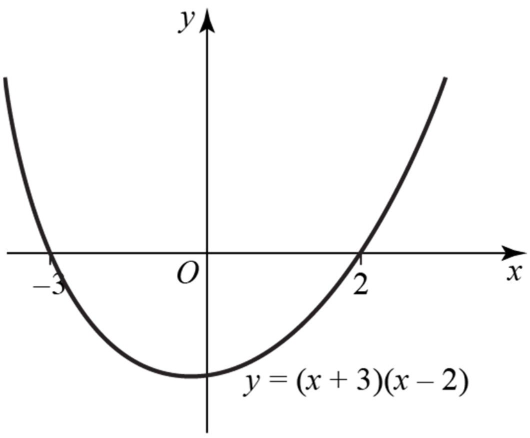
b $x^2+x\geqslant 6$ multiplying out left-hand side

$x^2+x-6\geqslant 0$

$(x+3)(x-2)\geqslant 0$ factorising

So the critical values are $x=2$ or -3

A sketch of $y=(x+3)(x-2)$ is



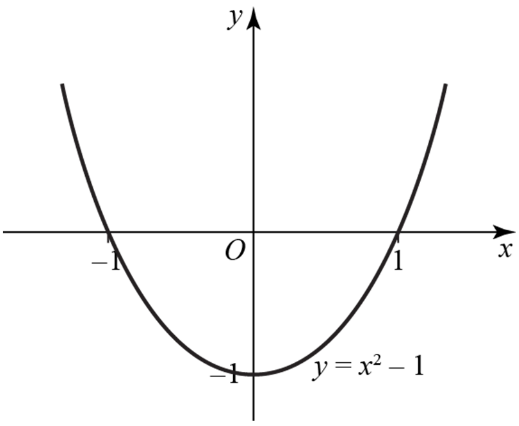
The solution corresponds to the section of the graph that is above or on the x-axis.
So the solution is $x\leqslant -3$ or $x\geqslant 2$

1 c $2 > x^2 + 1$ multplying both sides by $x^2 + 1$, you can do this because $x^2 + 1$ is always positive

$0 > x^2 - 1$

So the critical values are $x = \pm 1$

A sketch of $y = x^2 - 1$ is



The solution corresponds to the section of the graph that is below the x-axis.
So solution is $-1 < x < 1$

d To ensure multiplication by a positive quantity, multiply both sides by $(x^2 - 1)^2$

$\frac{2}{\cancel{(x^2-1)}} \times (x^2-1)^{\cancel{2}} > (x^2-1)^2$

$2(x^2 - 1) > (x^2 - 1)^2$

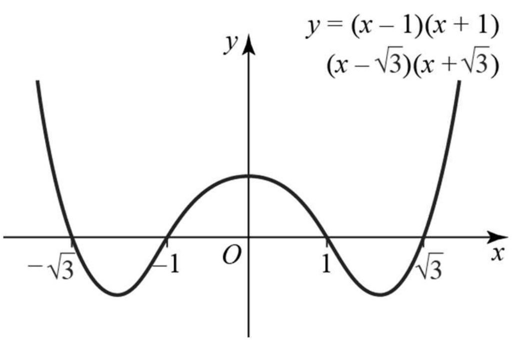
$0 > (x^2 - 1)\left((x^2 - 1) - 2\right)$

$0 > (x^2 - 1)(x^2 - 3)$

$0 > (x - 1)(x + 1)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

So the critical values are $x = \pm 1,\ x = \pm\sqrt{3}$

The curve $y = (x-1)(x+1)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$ is a quartic graph with positive x^4 coefficient, so the curve starts in the top left and ends in the top right and passes through $\left(-\sqrt{3}, 0\right)$, $(-1, 0)$, $(1, 0)$ and $\left(\sqrt{3}, 0\right)$. A sketch of the curve is



The solution corresponds to the section of the graph that is below the x-axis.
So the solution is $-\sqrt{3} < x < -1$ or $1 < x < \sqrt{3}$

1 e To ensure multiplication by a positive quantity, multiply both sides by $(x-1)^2$

$$\frac{x}{(x-1)} \times (x-1)^2 \leqslant 2x(x-1)^2$$

$$x(x-1) - 2x(x-1)(x-1) \leqslant 0$$

$$x(x-1)\big(1-2(x-1)\big) \leqslant 0$$

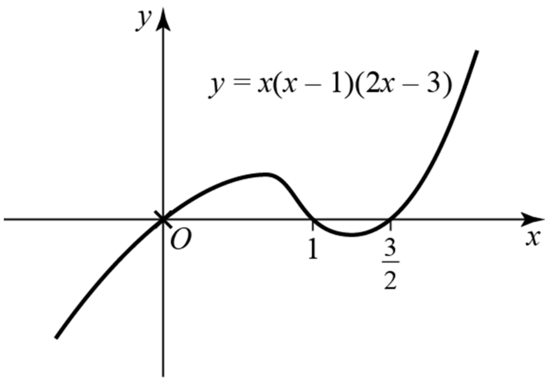
$$x(x-1)(-2x+3) \leqslant 0$$

$x(x-1)(2x-3) \geqslant 0$ multiplying by -1 so change the direction of the inequality

So the critical values are $x = 0, 1$ or $\frac{3}{2}$

The curve $y = x(x-1)(2x-3)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(0,0), (1,0)$ and $(\frac{3}{2}, 0)$.

A sketch of the curve is



The solution corresponds to the section of the graph that is above or on the x-axis, excluding $x = 1$ as the inequality is not defined for this value.

So the solution is $0 \leqslant x < 1$ or $x \geqslant \frac{3}{2}$

1 f To ensure multiplication by a positive quantity, multiply both sides by $(x+1)^2 x^2$

$$\frac{3}{(x+1)} \times (x+1)^2 x^2 < \frac{2}{x} \times (x+1)^2 x^2$$

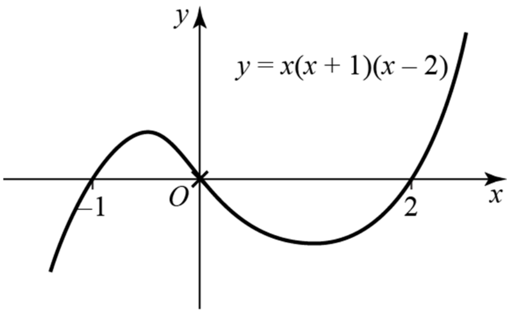
$$x(x+1)\big(3x-2(x+1)\big) < 0$$

$$x(x+1)(x-2) < 0$$

So the critical values are $x = 0, -1$ or 2

The curve $y = x(x+1)(x-2)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-1,0), (0,0)$ and $(2,0)$.

A sketch of the curve is



The solution corresponds to the section of the graph that is below the x-axis.
So the solution is $x < -1$ or $0 < x < 2$

g To ensure multiplication by a positive quantity, multiply both sides by $(x+1)^2(x-1)^2$

$$\frac{3}{(x+1)(x-1)} \times (x+1)^2 (x-1)^2 < (x+1)^2(x-1)^2$$

$$0 < (x+1)(x-1)\big((x+1)(x-1)-3\big)$$

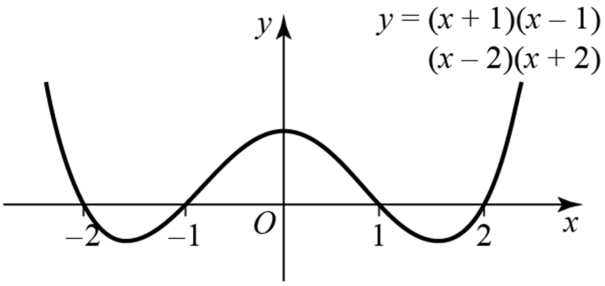
$$0 < (x+1)(x-1)(x^2-1-3)$$

$$0 < (x+1)(x-1)(x^2-4)$$

$$0 < (x+1)(x-1)(x-2)(x+2)$$

So the critical values are $x = \pm 1$ or ± 2

The curve $y = (x+1)(x-1)(x-2)(x+2)$ is a quartic graph with positive x^4 coefficient, so the curve starts in the top left and ends in the top right and passes through $(-2,0), (-1,0), (1,0)$ and $(2,0)$. A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution is $x < -2$ or $-1 < x < 1$ or $x > 2$

1 h To ensure multiplication by a positive quantity, multiply both sides by $x^2(x+1)^2(x-2)^2$

$$\frac{2}{x^2}\times x^2(x+1)^2(x-2)^2 \geqslant \frac{3x^2(x+1)^2(x-2)^2}{(x+1)(x-2)}$$

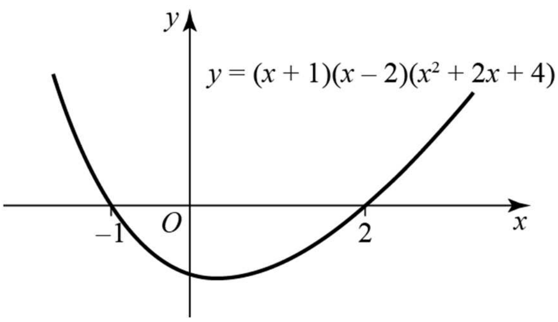
$$(x+1)(x-2)\left(2(x+1)(x-2)-3x^2\right)\geqslant 0$$

$$(x+1)(x-2)(-4-2x-x^2)\geqslant 0$$

$(x+1)(x-2)(x^2+2x+4)\leqslant 0$ multiplying by -1 so change the direction of the inequality

The discriminant of x^2+2x+4 is -12, so this quadratic has no real roots.
So the critical values are $x=2$ or -1

The curve $y=(x+1)(x-2)(x^2+2x+4)$ is a quartic graph with positive x^4 coefficient. A sketch of the curve is



The solution corresponds to the section of the graph that is below or on the x-axis, excluding those values of x for which the inequality is not defined, i.e. $x=-1, 0$ or 2.
So the solution can be expressed as either $-1<x<2 \quad x\neq 0$, or $-1<x<0$ or $0<x<2$

i To ensure multiplication by a positive quantity, multiply both sides by $(x-4)^2$

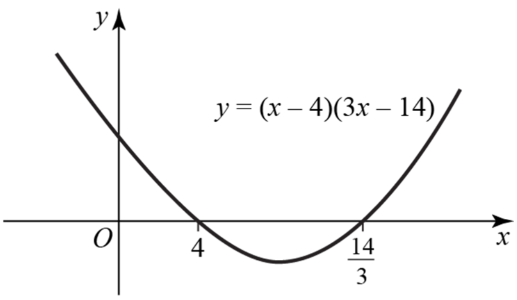
$$\frac{2}{x-4}\times(x-4)^2<3(x-4)^2$$

$$0<(x-4)\left(3(x-4)-2\right)$$

$$0<(x-4)(3x-14)$$

So the critical values are $x=4$ or $\frac{14}{3}$

The curve $y=(x-4)(3x-14)$ is a quadratic graph with positive x^2 coefficient. A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution is $x<4$ or $x>\frac{14}{3}$

1 j To ensure multiplication by a positive quantity, multiply both sides by $(x+2)^2(x-5)^2$

$$\frac{3}{(x+2)}\times(x+2)^2(x-5)^2 > \frac{1}{(x-5)}\times(x+2)^2(x-5)^2$$

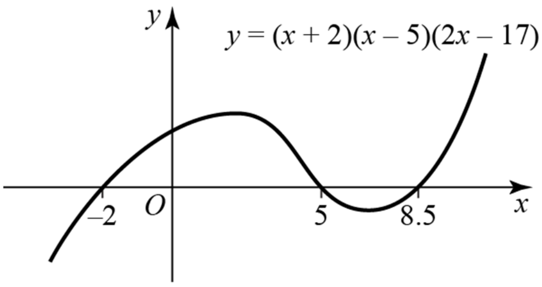
$$(x+2)(x-5)\big(3(x-5)-(x+2)\big)>0$$

$$(x+2)(x-5)(2x-17)>0$$

So the critical values are $x=-2$, 5 or $\frac{17}{2}$

The curve $y=(x+2)(x-5)(2x-17)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-2,0)$, $(5,0)$ and $(\frac{17}{2},0)$.

A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution is $-2<x<5$ or $x>\frac{17}{2}$

2 a To ensure multiplication by a positive quantity, multiply both sides by $(x+5)^2$

$$\frac{3x^2+5}{(x+5)}\times(x+5)^2 > (x+5)^2$$

$$(x+5)\big(3x^2+5-(x+5)\big)>0$$

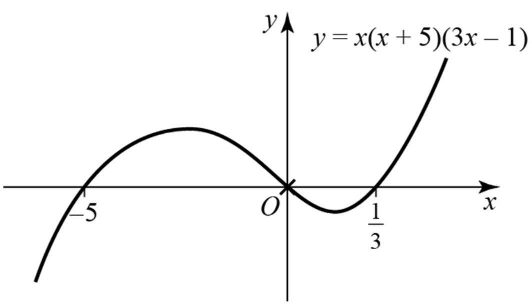
$$(x+5)(3x^2-x)>0$$

$$x(x+5)(3x-1)>0$$

So the critical values are $x=-5$, 0 or $\frac{1}{3}$

The curve $y=x(x+5)(3x-1)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-5,0)$, $(0,0)$ and $(\frac{1}{3},0)$.

A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution in set notation is $\{x: -5<x<0\}\cup\{x: x>\frac{1}{3}\}$

2 b To ensure multiplication by a positive quantity, multiply both sides by $(x-2)^2$

$\frac{3x}{\cancel{x-2}} \times (x-2)^{\cancel{2}} > x(x-2)^2$

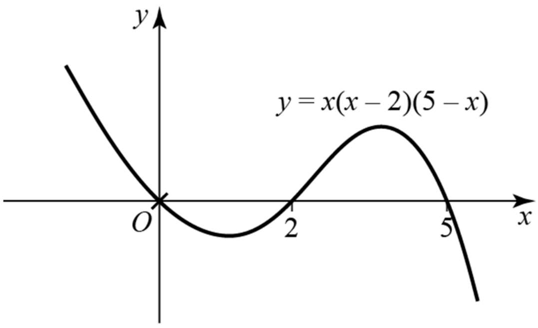
$x(x-2)\big(3-(x-2)\big) > 0$

$x(x-2)(5-x) > 0$

So the critical values are $x = 0, 2$ or 5

The curve $y = x(x-2)(5-x)$ is a cubic graph with negative x^3 coefficient, so the curve starts in the top left and ends in the bottom right and passes through $(0,0), (2,0)$ and $(5,0)$.

A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution in set notation is $\{x: x < 0\} \cup \{x: 2 < x < 5\}$

c To ensure multiplication by a positive quantity, multiply both sides by $(1-x)^2(2+x)^2$

$\frac{1+x}{\cancel{1-x}} \times (1-x)^{\cancel{2}}(2+x)^2 > \frac{2-x}{\cancel{2+x}} \times (1-x)^2(2+x)^{\cancel{2}}$

$(1-x)(2+x)\big((1+x)(2+x)-(2-x)(1-x)\big) > 0$

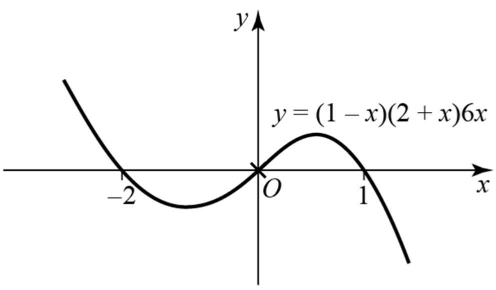
$(1-x)(2+x)(x^2+3x+2-(x^2-3x+2)) > 0$

$(1-x)(2+x)6x > 0$

So the critical values are $x = -2, 0$ or 1

The curve $y = (1-x)(2+x)6x$ is a cubic graph with negative x^3 coefficient, so the curve starts in the top left and ends in the bottom right and passes through $(-2,0), (0,0)$ and $(1,0)$.

A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution in set notation is $\{x: x < -2\} \cup \{x: 0 < x < 1\}$

2 **d** To ensure multiplication by a positive quantity, multiply both sides by $(x+1)^2$

$\frac{x^2+7x+10}{x+1} \times (x+1)^2 > (2x+7)\times(x+1)^2$

$(x+1)\left(x^2+7x+10-(2x+7)(x+1)\right) > 0$

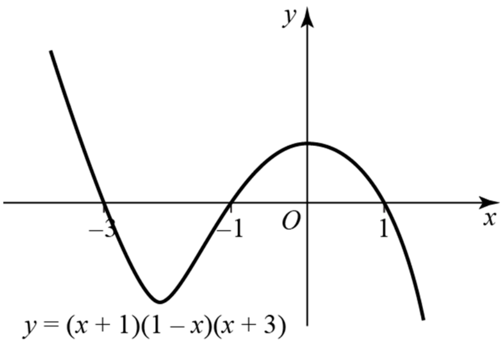
$(x+1)(x^2+7x+10-2x^2-9x-7) > 0$

$(x+1)(3-2x-x^2) > 0$

$(x+1)(1-x)(x+3) > 0$

So the critical values are $x = -3, -1$ or 1

The curve $y=(x+1)(1-x)(x+3)$ is a cubic graph with negative x^3 coefficient, so the curve starts in the top left and ends in the bottom right. A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution in set notation is $\{x: x < -3\} \cup \{x: -1 < x < 1\}$

e Multiply both sides by x^2

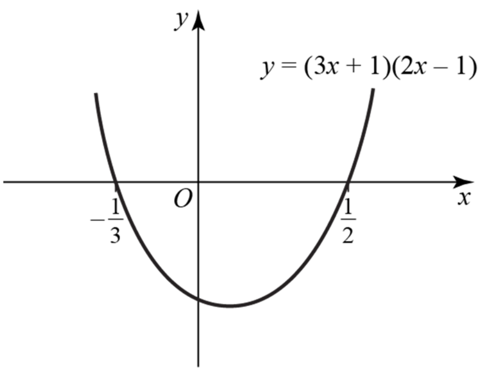
$x+1 > 6x^2$

$6x^2 - x - 1 < 0$

$(3x+1)(2x-1) < 0$

So the critical values are $x = -\frac{1}{3}$ or $\frac{1}{2}$

A sketch of the quadratic curve $y=(3x+1)(2x-1)$ is



The solution corresponds to the section of the graph that is below the x-axis, but excluding $x = 0$ as the inequality is not defined for this value
So the solution in set notation is $\{x: -\frac{1}{3} x < 0\} \cup \{x: 0 < x < \frac{1}{2}\}$

2 f Multiply both sides by $6(x+1)^2$

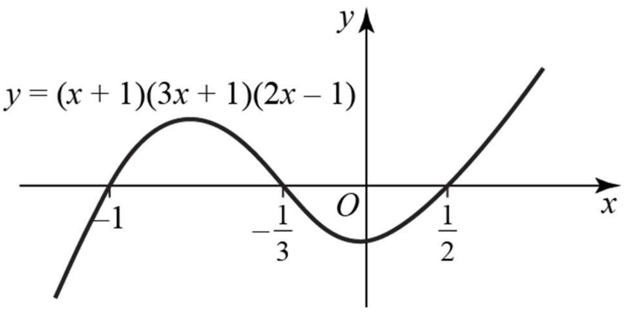
$\frac{6x^2}{x+1} \times (x+1)^2 > (x+1)^2$

$(x+1)\left(6x^2 - (x+1)\right) > 0$

$(x+1)(3x+1)(2x-1) > 0$

So the critical values are $x = -1, -\frac{1}{3}$ or $\frac{1}{2}$

The curve $y = (x+1)(3x+1)(2x-1)$ is a cubic. A sketch of the curve is



The solution corresponds to the section of the graph that is above the x-axis.
So the solution in set notation is $\{x: -1 < x < -\frac{1}{3}\} \cup \{x: x > \frac{1}{2}\}$

3 Multiply both sides by $(x+5)^2(x+4)^2$

$$\frac{2x+1}{\cancel{x+5}}\times(x+5)^{\cancel{2}}(x+4)^2<\frac{x+2}{\cancel{x+4}}\times(x+5)^2(x+4)^{\cancel{2}}$$

$$(x+5)(x+4)\big((2x+1)(x+4)-(x+2)(x+5)\big)<0$$

$$(x+5)(x+4)(2x^2+9x+4-x^2-7x-10)<0$$

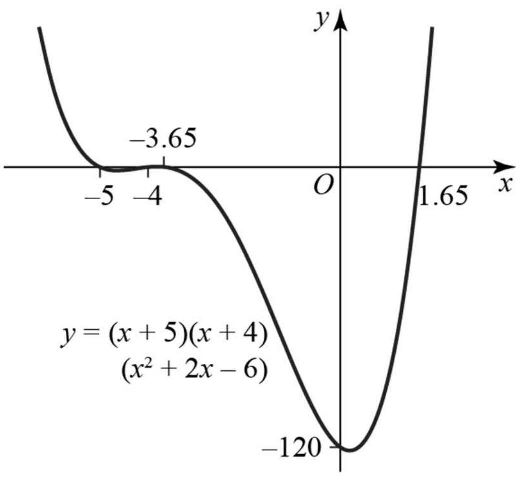
$$(x+5)(x+4)(x^2+2x-6)<0$$

To find critical values solve $x^2+2x-6=0$ using the quadratic formula:

$$x=\frac{-2\pm\sqrt{4+4\times6}}{2}=\frac{-2\pm\sqrt{4+24}}{2}=\frac{-2\pm\sqrt{28}}{2}=\frac{-2\pm2\sqrt{7}}{2}=-1\pm\sqrt{7}$$

So the critical values are $x=-5,-4$ or $-1\pm\sqrt{7}$

The graph of $y=(x+5)(x+4)(x^2+2x-6)$ is a quartic graph with positive x^4 coefficient that passes through the x-axis at $(-5,0),(-4,0),\left(-1-\sqrt{7},0\right)$ and $\left(-1+\sqrt{7},0\right)$. A sketch of the curve is



The solution corresponds to the section of the graph that is below the x-axis.

So the solution is $-5<x<-4$ or $-1-\sqrt{7}<x<-1+\sqrt{7}$

In set notation this is $\{x:-5<x<-4\}\cup\{x:-1-\sqrt{7}<x<-1+\sqrt{7}\}$

4 Multiply both sides by $(2x+1)^2(x-3)^2$

$$\frac{x}{(2x+1)} \times (2x+1)^2(x-3)^2 < \frac{1}{x-3} \times (2x+1)^2(x-3)^2$$

$$(2x+1)(x-3)\big(x(x-3)-(2x+1)\big) < 0$$

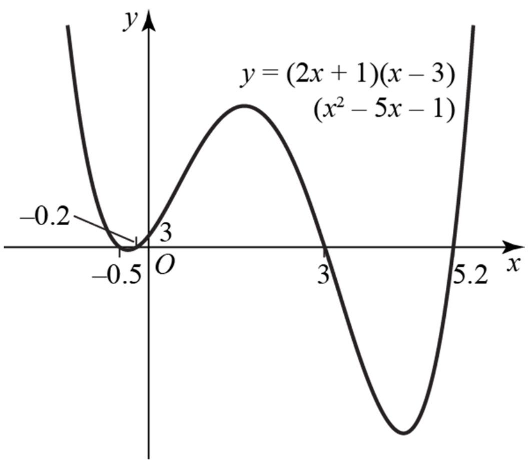
$$(2x+1)(x-3)(x^2-5x-1) < 0$$

To find critical values solve $x^2-5x-1=0$ using the quadratic formula:

$$x = \frac{5 \pm \sqrt{25-4\times(-1)}}{2} = \frac{5\pm\sqrt{29}}{2}$$

So the critical values are $x = -\frac{1}{2}, 3$ or $\frac{5\pm\sqrt{29}}{2}$

The graph of $y=(2x+1)(x-3)(x^2-5x-1)$ is a quartic graph with positive x^4 coefficient that passes through the x-axis at $\left(-\frac{1}{2},0\right), \left(\frac{5-\sqrt{29}}{2},0\right)$, $(3,0)$ and $\left(\frac{5+\sqrt{29}}{2},0\right)$. A sketch of the curve is



The solution corresponds to the section of the graph that is below the x-axis.

So the solution is $\left\{x: -\frac{1}{2} < x < \frac{5-\sqrt{29}}{2}\right\} \cup \left\{x: 3 < x < \frac{5+\sqrt{29}}{2}\right\}$

5 **a** The student did not square the denominators before multiplying, but has multiplied both sides by $x(3x+4)$. This expression can have negative values, which would not preserve the inequality.

b Multiply both sides by $x^2(3x+4)^2$

$\frac{x}{(3x+4)} \times x^2(3x+4)^2 < \frac{1}{x} \times x^2(3x+4)^2$

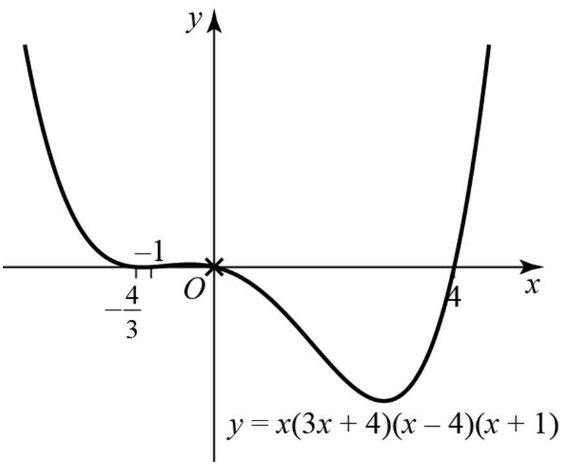
$x(3x+4)\left(x^2-(3x+4)\right)<0$

$x(3x+4)(x^2-3x-4)<0$

$x(3x+4)(x-4)(x+1)<0$

The critical values are $x=-\frac{4}{3}, -1, 0,$ or 4

The graph of $y=x(3x+4)(x-4)(x+1)$ is a quartic graph with positive x^4 coefficient that passes through the x-axis at $(-\frac{4}{3},0), (-1,0), (0,0)$ and $(4,0)$. A sketch of the curve is



The solution corresponds to the section of the graph that is below the x-axis.

So the solution is $-\frac{4}{3}<x<-1$ or $0<x<4$

In set notation this is $\{x: -\frac{4}{3}<x<-1\} \cup \{x: 0<x<4\}$

6 Consider each inequality in turn. First $\frac{4}{x} < x$

Multiply both sides by x^2

$\frac{4}{\cancel{x}} \times x^{\cancel{2}} < x \times x^2$

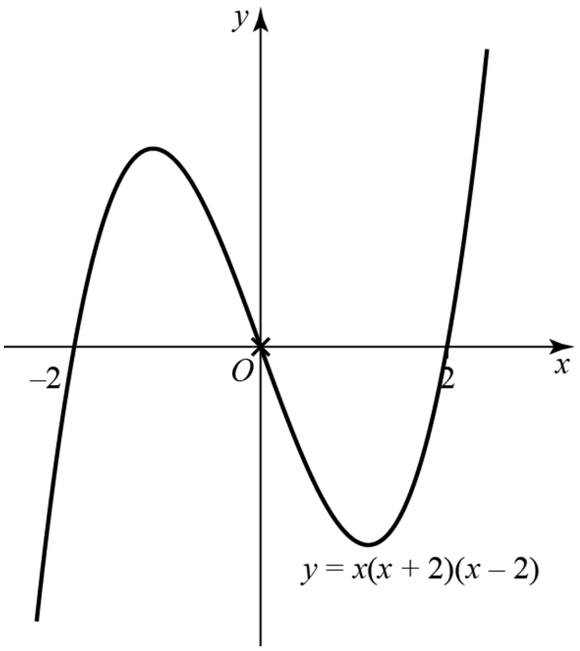
$x^3 - 4x > 0$

$x(x^2 - 4) > 0$

$x(x+2)(x-2) > 0$

So the critical values are $x = -2, 0$ or 2

A sketch of $y = x(x+2)(x-2)$ is



The solution corresponds to the section of the graph that is above the x-axis.

So the solution is $\{x: -2 < x < 0\} \cup \{x: x > 2\}$

Now consider $x < \frac{1}{2x+1}$

Multiply both sides by $(2x+1)^2$

$x(2x+1)^2 < \frac{1}{\cancel{(2x+1)}} \times (2x+1)^{\cancel{2}}$

$(2x+1)\left(x(2x+1) - 1\right) < 0$

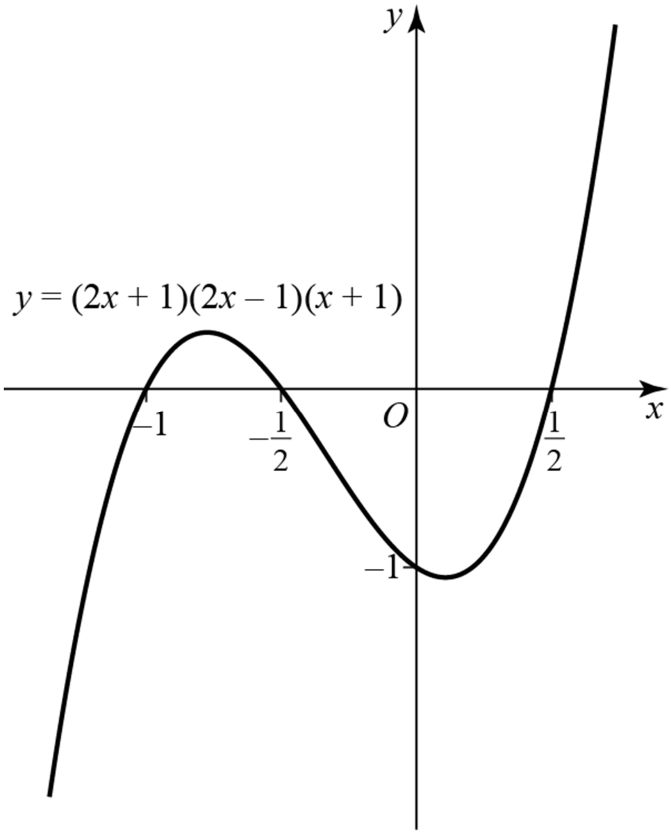
$(2x+1)(2x^2 + x - 1) < 0$

$(2x+1)(2x-1)(x+1) < 0$

So the critical values are $x = -1, -\frac{1}{2}$ or $\frac{1}{2}$

A sketch of $y = (2x+1)(2x-1)(x+1)$ is

6 continued



The solution corresponds to the section of the graph that is below the x-axis.
So the solution is $\{x: x<-1\} \cup \{x: -\frac{1}{2}<x<\frac{1}{2}\}$

For the inequality $\frac{4}{x}<x<\frac{1}{2x+1}$ to be satisfied, both solutions should hold.

So the solution $=\left(\{x: -2<x<0\} \cup \{x: x>2\}\right) \cap \left(\{x: x<-1\} \cup \{x: -\frac{1}{2}<x<\frac{1}{2}\}\right)$

$= \{x: -2<x<-1\} \cup \{x: -\frac{1}{2}<x<0\}$

Challenge

As $e^x > 0$, multiply both sides by $(1-e^x)^2 e^x$

$$\frac{1}{(1-e^x)} \times (1-e^x)^2 e^x < \frac{1}{e^x} \times (1-e^x)^2 e^x$$

$$(1-e^x)\left(e^x - (1-e^x)\right) < 0$$

$$(1-e^x)(2e^x - 1) < 0$$

$(e^x - 1)(2e^x - 1) > 0$ multiplying by -1 and switching the direction of the inequality

The critical values are $e^x = \frac{1}{2}$ or 1

To find out where the equality holds, use test values in each region

$e^x = \frac{1}{4} \Rightarrow (e^x - 1)(2e^x - 1) = -\frac{3}{4} \times -\frac{1}{2} = \frac{3}{8}$, which is greater than 0

$e^x = \frac{3}{4} \Rightarrow (e^x - 1)(2e^x - 1) = -\frac{1}{4} \times \frac{1}{2} = -\frac{1}{8}$, which is less than 0

$e^x = \frac{3}{2} \Rightarrow (e^x - 1)(2e^x - 1) = \frac{1}{2} \times 2 = 1$, which is greater than 0

So the inequality holds for $e^x < \frac{1}{2}$ or $e^x > 1$

To express the solution in terms of x, take logs of both sides, $e^x = \frac{1}{2} \Rightarrow x = \ln\frac{1}{2}$, $e^x = 1 \Rightarrow x = \ln 1$

So the solution is $x < \ln\frac{1}{2}$ or $x > \ln 1$